

On the Generalized Exclusion Statistics ¹

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Abstract

We review the principal steps leading to drive the wave function $\psi_{\{k_1, k_2, \dots, k_N\}}(1, 2, \dots, N)$ of a gaz of N identical particle states with exotic statistics. For spins $s = 1/M \bmod(1)$, we show that the quasidetterminant conjectured in [19], by using $2d$ conformal field theoretical methods, is indeed related to the quantum determinant of noncommutative geometry. The q-number $[N]! = \prod_{n=1}^N (\sum_{j=0}^{N-1} q^j)$ carrying the effect of the generalized Pauli exclusion principle, according to which no more than $(M - 1)$ identical particles of spin $s = 1/M \bmod(1)$ can live altogether on the same quantum state, is rederived in rigorous from the q-antisymmetry. Other features are also given.

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1 Introduction

Statistics is a quantum mechanical feature of identical particles which governs the quantum behavior of a collection of N particles in the large N limite. In the $(1 + 3)$ dimentional real world where experiments are available both at low and high energy physics, theory and experiments agree on the Pauli classification of particles into bosons and fermions respectively of integer and half odd integer spins. Outside the real world where experiments are not yet available (except for very special condensed matter systems [1]), there are only theoretical predictions. For higher dimensions, representations group theory predict that like in $(1 + 3)d$ world, the Pauli exclusion principle should usually hold. In two dimensions however, theory predicts that one can go beyond the Pauli classification although one should worry about locality if one approach exotic statistics by using $2d$ quantum field theoretical methods [2].

As far as exotic statistics is concerned and though it was considered in some occasions in the past [3], we think that its right development was initiated only in the begining of the eighties, especially after the work of Belavin et al [4] on $2d$ conformal field theory (CFT). Since this development, exotic statistics has been attracting interest of attension as they are involved in different areas of theoretical physics [5,6,7]. Besides its natural generalisation of the Bose and Fermi statistics, one recalls for instance the role its plays in conformal minimal models and thier integrable deformations [6]. Exotic statistics interpolating between bosons and fermions plays also a centrale role in anyon superconductivity, fractional supersymmetry [8,9], $U_q(sl(2))$ and affine $U_q(\hat{sl}(2))$ quantum groups representations [7] and non commutative geometry [10]. The latter is getting more and more importance in superstring theory, especially after the development made by Connes et al [11] in the context of the compactification of the matrix models of M theory [12], and too recently in type IIB on $AdS_5 \times S^5/Z_3$ and $4d$ $N = 1$ supersymmetric $su(N)^3$ gauge theory with bifundamental matters [13]. Applications of exotic statistics are then multiple as it is involved in many areas of modern physics. One of the these applications which has been considered recently and which be the subject of this paper is that consediring the thermodynamic properties of a quantum ideal gas of particles obeying exotics statistics. In this contex there are different approches dealing with such kind of statistical systems. One of these approaches is the interesting one initiated by Haldane [14] and developed by Wu [15], where it was shown that the mean occupation number

$\langle n_k \rangle$ of states of definite energy ϵ_k is given by

$$\langle n_k \rangle = \frac{1}{w(e^{\beta(\epsilon_k - \mu)}) - \alpha}. \quad (1)$$

In this equation, $\beta = 1/(kT)$ and $w(\xi)$ is a function obeying the following identity

$$w(\xi)^\alpha [1 + w(\xi)]^{1-\alpha} = \xi = e^{\beta(\epsilon_k - \mu)}. \quad (2)$$

The parameter α describes the interpolating statistics between bosons ($\alpha = 0$ and $w(\xi) = \xi$) and fermions ($\alpha = 1$ and $w(\xi) = \xi - 1$). An alternative approach dealing with ideal gases of identical particles of exotic statistics was also considered by one of us in [16] and developed later in [17,18,19]. In this approach, one considers particle states of spin $1/M \pmod{1}$; $M \geq 2$ as basic objects on the same footing as bosons and fermions associated with $M = \infty$ and $M = 2$ respectively. This consideration is essentially motivated by the fact that particles of fractional spin ($1/M$) appears as well definite state in $2d$ fractional supersymmetry and in the periodic representations of the quantum group $U_q(\mathfrak{sl}(2))$ with q a root of unity i.e $q = \exp(\frac{2i\pi}{M})$ [8,9,20]. One of the remarkable results of our way of doing is the statement of the principle governing particles of spin $1/M \pmod{1}$. Such principle extends the standard Pauli exclusion and tells that no more than $(M - 1)$ particles of spin $1/M \pmod{1}$ can live altogether on the same quantum state. In connection with this results, it was shown in [16] that the mean occupation number $\langle n_k \rangle$ describing the quantum behaviour of such exotic statistics is given by

$$\langle n_k \rangle = \frac{1}{e^{-\beta(\epsilon_k - \mu)} - 1} - \frac{M}{e^{-\beta M(\epsilon_k - \mu)} - 1}. \quad (3)$$

Taking $M = \infty$ and $M = 2$ in eq.(3), one discovers the usual Bose-Einstein and Fermi-Dirac distributions respectively. Eq.(3) was first established in [16] and at that time the author of [16] had no idea on the Haldane proposal. The search for the distribution (3) was motivated only by studies on parafermions á la Zamolodchikov and Fateev [21], fractional superstrings á la Tye et al [22] and integrable deformation of conformal field theory [23]. It should be noted here that in our analysis we start from the idea that particle states of fractional spin exist as individual states and thier effects should be a priori visible if $2d$ experiments are available. In the Haldane-Wu analysis however, one starts from the idea that exotic statistics appears only as an interpolating statistics between bosons and fermions so that quantum states

obeying generalized statistics play a secondary role only. We will not discuss here these two ways of viewing things; what we will do rather is to look for common features in both analysis. For example one could like to know whether there exists a link between eqs.(1) and (3). What is the relation between the statistical weights in both approaches and how exclusion statistics may be stated. In this paper we would like also to answer the open question of [19] concerning the link between the quasideterminant Δ_q used there and the quantum determinant det_q of non commutative geometry.

The presentation of this paper is as follows: In section 2, we study the generalized statistical weight density and discuss the link between eqs.(1) and (3) and the relation between the Haldane interpolating parameter α and the spin. In section 3, we review the derivation of the wave function of the gas of N identical particles of spin $(1/M) \bmod 1$ by using non commutative geometric methods. Actually this section answers the open question arised in [19]. In section 4 we study the generalized Pauli exclusion principle for particles of spin $(1/M) \bmod 1$; $M \geq 2$ by using q-antisymmetry. In section 5, we give our conclusion.

2 The generalized statistical weight

First of all note that there are different ways of introducing generalized statistics. For the gas of N particles with exclusion statistics we are interested in here, there are at least three ways of doing. The first way we will review briefly in this section, and which is due to Haldane, is based on the following combinatorial formula $W(\alpha)$ giving the number of accessible states of the gas of N particles occupying a group of G states

$$W(\alpha) = \frac{[G + (\alpha - 1)(N - 1)]!}{N![G - \alpha(N - 1)]!}, \quad (4)$$

In this equation, α is an interpolating parameter $0 \leq \alpha \leq 1$ parametrising the generalized statistics. The second way of doing is that considered in [16,17,18]. It deals with quantum states with spin $(1/M) \bmod 1$ and it is based on the conjecture that no more than $(M - 1)$ particles can live altogether on the same quantum state. The third way of doing was considered recently in [24]; it starts from postulating a formula for the wave function of the gas interpolating between the wave function of bosons and fermions by

following the reasoning used for the densities eq.(4). For spin $(1/M) \bmod 1$, the obtained wave function coincide with that given in [19]. Let us now turn to explor the commun denominator between the Haldane and Wu approache and ours. First observe that eq.(4) may be rewritten as

$$W(\alpha) = \frac{(x+y)!}{x!y!} = \frac{\Gamma(x+y+2)!}{\Gamma(x+1)\Gamma(y+1)}, \quad (5)$$

where $x = N$; $y = y(\alpha) = G - \alpha N - (1 - \alpha)$ and $\beta(x, y)$ is the well known digamma special function. Note that $y(\alpha)$ is just the linear interpolation between the pionts $y_B = y(0) = G - 1$ and $y_F = y(1) = G - N$ associated with the spin $s = 0 \bmod 1$ and $s = 1/2 \bmod 1$ respectively. this features as well as properties of the special function $\beta(x, y)$, in particular its integral definition, shows that the Haldane interpolating parameter α and the spin s are related in the large N limit as

$$\alpha = \frac{N(1 - 2s)}{N - 2}. \quad (6)$$

For the special case of fractional spins we are considering in this paper that is for spin $(1/M) \bmod 1$, eq.(6) reduces to

$$\alpha = \frac{N(M - 2)}{M(N - 2)}. \quad (7)$$

Taking $M = 2$ and $M = N \rightarrow \infty$, we discover respectively the values $\alpha = 0$ describing fermions and $\alpha = 1$ for bosons. Moreover putting back eq.(7) into eq.(4), one gets the generalized statistical weight for spin $(1/M) \bmod 1$. Actually this identity constitus the first bridge between our way of doing and the Haldane analysis. The second bridge we want to give concerns the link between eqs.(1) and (3). Both of these distributions may be obtained by computing the partition function \mathcal{Z} of the gas. Using the generalized Pauli exclusion principle according to which no more than $(M - 1)$ particle can live altogether on the same quantum state. It is not difficult to check that the grand partition function \mathcal{Z} is given by

$$\mathcal{Z} = \prod_{j \geq 0} \mathcal{Z}_j, \quad \mathcal{Z}_j = \frac{1 - \exp(-\beta M(\epsilon_j - M))}{1 - \exp(-\beta(\epsilon_j - M))}. \quad (8)$$

Eq.(8) extends the usual Bose and Fermi gas partition functions and leads to the mean occupation number of eq.(3). In the Haldane and Wu analysis,

one uses the following property of the single level partition function $\mathcal{Z}(\xi)$

$$\xi(\mathcal{Z}^\alpha(\xi) - \mathcal{Z}^{(\alpha-1)}(\xi)) = 1 \quad (9)$$

in order to calculate the mean number $\bar{n} = \xi \frac{\partial}{\partial \xi} \log \mathcal{Z}(\xi)$. In doing so, one discovers that the mean number \bar{n} obeys exactly eqs.(1-2). It should be noted that the explicit form of the quantum distribution (1) depends on the solving of eqs.(2). In [15], some special solutions were obtained. These concerns the Bose and Fermi distributions but also what is called there semions associated with $\alpha = 1/2$ and having as mean number

$$\bar{n}_j(\alpha = 1/2) = \frac{1}{[\frac{1}{4} + \exp 2\beta(\epsilon_j - \mu)]^{1/2}}. \quad (10)$$

For general α 's, the solution of eq.(2) is still missing. However if one requires that at most $(M - 1)$ particles can be put per site with probabilities 1, one discovers eq.(3).

3 The generalized wave function

In [17,19], an interpolating formula for the $\psi_{\{k_1, k_2, \dots, k_N\}}(1, 2, \dots, N)$ wave function of a system of N identical spin $(1/M) \bmod 1$ particles was proposed. It is shown that ψ is a kind of quasideterminant interpolating between the usual determinant and the permanent of fermions and bosons respectively. One of the remarkable features of the quasideterminant as defined in [18], see also [17], is the generalized Pauli exclusion principle manifests itself through the identity

$$[M]! = (1 + q)(1 + q + q^2)(1 + q + q^2 + \dots + q^{M-1})[1][2]\dots[M], \quad (11)$$

which vanishes identically for the roots $q = \exp(\frac{2i\pi}{L})$ with $L = 2, 3, \dots, M$. For $M = 2$, one recovers the Pauli exclusion principle and for $M = \infty$ we get the Bose condensation. For generic values of M , $2 \leq M \leq \infty$, one has just what we have been calling the generalized Pauli exclusion principle. The obtention of eq.(11) was in fact expected since, on one hand it includes the q-fermionic number $Tr q^F$ which reads for spin $(1/M) \bmod 1$ particles states as follows

$$Tr q^F = 1 + q + q^2 + \dots + q^{M-1} \quad (12)$$

where $q^F = \exp(\frac{2i\pi}{M})$. this equation extends the usual fermionic number and was shown in [8] to characterise the generalized Pauli exclusion principle. On the other hand, the introduction of the quasideterminant was motivated by earlier works on deformation of the conformal field theory in particular the $SU_k(2)/U(1)$ WSW model where we have the following identity

$$(z_1 - z_2)^{h_1+h_2-h} \phi_m^{j_1}(z_1) \phi_m^{j_2}(z_2) = (z_2 - z_1)^{h_1+h_2-h} \phi_m^{j_2}(z_2) \phi_m^{j_1}(z_1). \quad (13)$$

Eqs.(13) gives the short distance of the $2d$ fields operators $\phi_m^{j_1}(z_1)$ and $\phi_m^{j_2}(z_2)$ of conformal weights h_1 and h_2 respectively; j is the $SU_k(2)$ isospin taking values in the range $0 \leq j \leq \frac{k}{2}$; $-j \leq m \leq j$ is the $U(1)$ Cartan charge and h is a given highest weight which can be read from the fusion algebra of primary fields. For more details see [22]. Dividing both sides of eq.(12) by $(z_1 - z_2)^{h_1+h_2-h}$, one gets a q -antisymmetric identity of deformation parameter $q = \exp \pm i\pi(h_1 + h_2 - h)$. On the other hand, it was suggested in [19] that eq.(11) and more generally the quasideterminant Δ_q should be derived in a rigorous way by using non commutative geometric methods. However in trying to establish this connection, we were faced in the above mentioned works to a major difficulty which we had not solved at that time. The problem is that the quasideterminant Δ_q uses only commutative \mathcal{C} -number exactly as the wave function $\psi_{k_j}(j)$ of the individual particles whereas the quantum determinant \det_q is mainly based on a non commutative algebra. In this section we want to complete this result by establishing the link between Δ_q of [19] and the quantum determinant of non commutative geometry. To start let us recall briefly that the wave function ψ of the gas (enclosed within a container volume V) is given by

$$\psi = \Delta_q(\psi_{k_1}(1), \psi_{k_2}(2), \dots, \psi_{k_N}(N)), \quad (14)$$

which for later use, we rewrite it as

$$\psi = \Delta_q(\psi_j^N). \quad (15)$$

In this equation $\psi_{k_j}(j)$ stands for the wave function of the individual particle j . ξ_j denote collectively all the coordinates of the j -th particle namely its two position coordinates (τ, σ) and its spin s . k_j is an index labeling its possible quantum states, in particular the energy ϵ_{k_j} the momentum P_{k_j} of the particle and the orientation of the spin. Finally the quasideterminant is roughly speaking given by

$$\Delta_q(\psi_{k_j}(j)) = \sum_{n=0}^{\frac{N(N-1)}{2}} q^n P_n(\psi_{k_j}(j)), \quad (16)$$

where $P_n(\psi_{k_j}(j))$ are homogenous polynomials in the individual wave functions. Details are given in [19]. Next, we introduce the q -deformed Grassman algebra generated by the system $\{\theta_1, \theta_2, \dots, \theta_N\}$ satisfying

$$\begin{aligned}\theta_i^2 &= 0 \\ \theta_i \theta_j &= q \theta_j \theta_i, \quad i < j \\ \theta_i \theta_j &= \frac{1}{q} \theta_j \theta_i, \quad i > j,\end{aligned}\tag{17}$$

and consider the space of generalized wave function $\psi(j)$ defined as

$$\psi(j) = \sum_{i=1}^N \psi_{k_j}(j) \theta_i.\tag{18}$$

In eq.(17-a), $\psi_{k_j}(j)$ are just the individual wave functions appearing in eq.(14). They may be viewed as \mathcal{C} -valued sections of a kind of a deformed N -dimensional one forms. For convenience, we prefer to rewrite eq.(17-a) as

$$\psi(j) = \sum_{i=1}^N \psi_{ij}(j) \theta_i = [\psi][\theta],\tag{19}$$

where we have replaced k_i by just the index i . Now, using eq.(16), one can work out the quantum determinant of $[\psi]$ by computing the generalized volume V given by the product of all the superwave functions $\psi(j)$, $j = 1, \dots, N$

$$V = \prod_{j=1}^N \psi(j) \equiv \det_q[\psi] \cdot \prod_{j=1}^N \theta_j.\tag{20}$$

Straightforward calculations show that V reads as

$$V = \Delta_q[\psi_{k_j}(j)] \prod_{i=1}^N \theta_i,\tag{21}$$

where $\Delta_q[\psi_{k_j}(j)]$ is exactly the quasideterminant we have introduced in [18] and which we can define also as

$$\Delta_q[\psi] = \sum_{p \in S_N} q^{i(p)} \psi_{p(1)}(1) \psi_{p(2)}(2) \dots \psi_{p(N)}(N).\tag{22}$$

In eq.(22) the summation is carried over all the permutations p of the particles and $i(p)$ is the number of inversions of a given p . $i(p)$ is just the minimal length of p . For details see [24], some useful features of eq.(20) will be considered in what follows.

4 Generalized Pauli exclusion principle

We begin by recalling that the Pauli exclusion principle manifests itself in different but equivalent ways. In the language of wave functions, the Pauli exclusion principle for fermions is carried by antisymmetry ensuring the vanishing of the Slater determinant whenever two fermions are in the same quantum state. In this section we want to show that this feature extends to particles with spin $(1/M) \bmod 1$ satisfying a generalized Pauli exclusion principle. We will show in particular that this exclusion statistics is carried by a q antisymmetry ensuring the vanishing of the wave function $\psi_{\{k_1, k_2, \dots, k_N\}}(1, 2, \dots, N)$ whenever M particles are in the same quantum state. To that purpose, let us first introduce the following convention notation

$$\begin{aligned} \prod_{j=1}^N \psi(j) &\equiv \det_q[\psi^N] \cdot \prod_{n=1}^N \theta_n \\ \prod_{j=1}^N \psi(j) &\equiv \det_q[\psi_1^N] \cdot \prod_{n=1}^N \theta_n \\ \prod_{j=1}^N \psi(j) &\equiv \det_q[\psi_i^N] \cdot \prod_{n=1}^N \theta_n, \end{aligned} \quad (23)$$

In eq.(23) $[\psi_i^{(N-1)}]$, $j \neq i$ is just the i -th $(N-1 \times N-1)_{(i,i)}$ minor of $[\psi]$. Next using eqs.(16-17), it is not difficult to check that the above quantum determinant $\det[\psi^N]$ and those of its minors $\det(\psi^{(N-1)})$ are related as

$$\det_q(\psi_i^{(N-1)}) = \sum_{j=1}^N q^{j-1} \psi_{k_N}(j) \det_q(\psi_j^{(N-1)}), \quad (24)$$

or equivalently by expanding along the n -th row

$$\det_q(\psi_i^{(N)}) = \sum_{j=1}^N q^{N-j} \psi_{k_N}(j) \det_q(\psi_j^{(N-1)}). \quad (25)$$

Note that eqs.(22) are just the generalisation of the Sarrus decomposition of the determinant. These equations which were conjectured in [18,19] are as the basis of the derivation of the wave function for identical particles of spin $(1/M) \bmod 1$. Note also that in the case where the $\psi_{k_i}(j) = \psi_{k_i}$ for all values of $j = 1, \dots, N$; eq.(22-a) factorizes as

$$\det_q(\psi) = [N]! \psi_{k_1} \psi_{k_2} \dots \psi_{k_N}, \quad (26)$$

where $[N]!$ is as in eq.(11). Actually eq.(26) was first derived in [18] by using a different methods. It vanishes not only for $M = 2$ ($q = 1$) as required by

the Pauli exclusion principle but also for all values of M lying between two and N . This means that the wave function $\psi_{\{k_1, k_2, \dots, k_N\}}(1, 2, \dots, N)$ vanishes identically whenever M collons of the quantum determinant $\det_q(\psi^N)$ are equal. In order words, this eq.(26) means that no more than $(M-1)$ identical particle of spin $(1/M) \bmod 1$ can live altogether on the same quantum state. This states then the generalized Pauli exclusion principle conjectured in [16].

5 conclusion

In this paper we have studied features of generalized exclusion statistics by using different methods. We have also given the links between technics of statistical mechanics dealing with exotic statistics and our way of doing inspired from $2d$ conformal field theoretical methods. After a bref description on the ways one can follows to deal with a gas of identical particles with exotic spins, we show that the properties of the Haldane combinatorial formula eq.(4) are exactly those of the well known digamma $\beta(x, y)$

$$\beta(x, y) = \int_0^1 t^{(x-1)} (1-t)^{(y-1)} dt. \quad (27)$$

Using standard features of $\beta(x, y)$, in particular the above integral definition, we have derived the link between the Haldane interpolating parameter α eq.(4) and the exotic spin s of the particles of the gas as shown on eqs.(6-7). We have also established the link between the quantum distributions eqs.(1) and (3). We have shown that our formula (3) is also a solution of the Wu system (1) corresponding to the limiting case where at most $(M-1)$ particles can be put per state with probabilities 1. We have also answered an open question rised in [18,19] asking for the link between our quasideterminant giving the wave function of the gas and the quantum determinant of non commutative geomerty. We have derived this link rigourously from q -anticommutativity. As a matter of facts we have derived some remarkable quantities which, we belive, has much to do with the generalized exclusion statistics. For spin $(1/M) \bmod 1$, we can say that we have now a formula carrying the effect of the generalized Pauli exclusion principle namely

$$[N]! = (1+q)(1+q+q^2) + \dots + (1+q + \dots + q^{M-1}) + \dots + (1+q + \dots + q^{M-1}). \quad (28)$$

This equation includes an other quantity which was suggested in [8] to describe this generalized principle. This concerns the q -fermionic number operator $Tr(q^F)$ eq.(12).

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References

- [1] B.I.Halperin, Phys. Rev. Lett. 52 (1984) 1583,
J.Frohlich, U.S.Studer and E.Thiram, in Proc. Conf. On three levels
Lauven, (1993), eds M.Anes et al.
- [2] I.Benkaddour and E.H.Saidi, Fractional supersymmetry as a matrix
model, Class. Quant. Grav. 16(1999) 1793.
- [3] J.M.Leinaas and J.Myrheim, Nuovo Cimenta B37 (1977) 1.
- [4] A.Belavin, A.M.Polyakov and A.B.Zamolodchikov, Nucl. Phys. B241
(1984) 333.
- [5] See for example, F.Wilczek, Fractional statistics and anyon super-
conductivity. (World scientific publishing) Singapou (1990) and refs
therein.
- [6] G.Musardo, Phys. Rep. 218 (1992) and refs therein.
- [7] L.D.Fadees, N.Yu.Reshetikhin and L.A. Takhtajan, Algebra analysis
1 (1987) 178.
M.Jimbo, T.Miwa and M.Odake, Lett. Math. Phys. 14 (1987) 123;
Comm. Math. Phys. 116 (1988) 508.
- [8] E. H.Saidi, M. B. Sedra and J. Zerouaoui, Class. Quant. Grav. 12
(1995) 1576.
E. H.Saidi, M. B. Sedra and J. Zerouaoui, Class. Quant. Grav. 12
(1995) 2705.
- [9] A.LeClair and C.Vafa, Nucl. Phys. B401 (1993) 413.
- [10] J.Wess, Quantum groups and quantum spaces, LMU-TPW 97-18,
A.Connes, Non commutative geometry, Academic Press (1994).
- [11] A.Connes, M.R.Douglas and A.Shwartz, JHEP 02 (1998) 03 hep-
th/9711162.
- [12] T.Banks, W.Fisher, S.H.Shenker and L.Susskind, Phys. Rev. D55
(1997) 5112, hep-th/96110043.

- [13] S.Gokov, M.Rangamani and E.Witten, Dibaryon, string and branes in ADS orbifold models, ASSNS-hep/98-93 ITEP-TH-30/98.
- [14] F.D.H.Haldane, Phys. Rev. Lett. 67 (1991) 37.
- [15] Y.S.Wu, Phys. Rev. Lett. 73, N7 (1994).
- [16] E.H.Saidi, Procceding de la journee de Physique Statistique; Avril (1995), FST Settat, Morocco.
- [17] . H.Saidi, ICTP preprint IC/95/299.
E.H.Saidi and J.Zerouaoui, Revue Marocaine des Sciences Physique N2 (2000).
- [18] J.Zerouaoui, PHD thesis, Rabat Morocco (1995) and Doctorat d'Etat, Rabat Morocco (2000).
- [19] M. Rachidi, E. H. Saidi and J. Zerouaoui, Phys. Lett. B409 (1997) 394.
- [20] H.Chakir, A.Elfallah and E.H.Saidi, Mod. Phys. Lett. A (1995) 2931 and Class. Quant. Grav. 14 (1997) 2049.
- [21] A.B.Zamolodchikov and V.A.Fateev, Sov. Phys. JETP 62 (1985) 215.
- [22] P.C.Argyres, K.R.Dienes and S.H.H.Tye, Comm. Math. Phys. 154 (1993) 471.
- [23] M.B.Sedra, PHD Thesis, Rabat Morocco (1993) and Doctorat d'Etat, Rabat Morocco (1995).
- [24] M. Rachidi, E. H. Saidi and J. Zerouaoui, submitted to JMP.